Categories:

A category $\mathcal{C}$ is

- A set of objects: $\mathrm{ob}\,\mathcal{C}$
- A set of morphisms for any two $X,Y \in \mathrm{ob}\,\mathcal{C}$: $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
- $\forall X \in \mathrm{ob}\,\mathcal{C}$ an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$
- A composition function $\forall X,Y,Z \in \mathrm{ob}\,\mathcal{C}$ $\circ : \mathrm{Hom}(Y,Z)\times \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(X,Z)$

which satisfies

- Associativity
- $\forall X,Y \in \mathrm{ob}\,\mathcal{C}$ & $f : X \longrightarrow Y$ $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$

For any category $\mathcal{C}$ we have the oposite category $\mathcal{C}^{op}$

- $\mathrm{ob}\,\mathcal{C}^{op} = \mathrm{ob}\,\mathcal{C}$
- $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$
- $g^{op} \circ f^{op} = (f \circ g)^{op}$

Turns around all the arrows.

Morphisms: $\mathcal{C}$ category, $X,Y \in \mathrm{ob}\,\mathcal{C}$, $f : X \longrightarrow Y$ is an

- isomorphism $\Leftrightarrow$ $\exists g : Y \longrightarrow X$ $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$
- endomorphism $\Leftrightarrow$ $X = Y$
- Automorphism $\Leftrightarrow$ $X = Y$ & f is an isomorphism

A groupoid is a category where all morphisms are isomorphisms i.e. have inverses a la groups.

A SubCategory: $\mathcal{D} \subseteq \mathcal{C}$ is a sub-set of objects & morphisms from $\mathcal{C}$

- closed under composition
- containing $\forall X \in \mathrm{ob}\,\mathcal{D}$ the morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$.

A subcategory $\mathcal{D}$ is full iff $\forall X,Y \in \mathrm{ob}\,\mathcal{D}$ $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ i.e. no morphisms are missing in $\mathcal{D}$.

Functors: A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of functions

- $F : \mathrm{ob}\,\mathcal{C} \longrightarrow \mathrm{ob}\,\mathcal{D}$
- $\forall X,Y \in \mathrm{ob}\,\mathcal{C}$ a function $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ such that
  - $F(\mathrm{id}_X) = \mathrm{id}_{FX}$
  - $F(g \circ f) = F(g) \circ F(f)$

The image of an isomorphism under a functor is an isomorphism. Moreover if $X \cong Y \Rightarrow FX \cong FY$.

A functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$ is called a contravariant functor $\mathcal{C} \longrightarrow \mathcal{D}$

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is full if $\forall X,Y \in \mathrm{ob}\,\mathcal{C}$ $F_{X,Y}$ is surjective, faithful if they are injective and fully faithful if bijective.

Natural Transformations: $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors. A natural transformation $\varphi : F \longrightarrow G$ is a collection of functions $\forall X \in \mathrm{ob}\,\mathcal{C}$, $\varphi_X : FX \longrightarrow GX$ such that $\forall f \in \mathrm{Hom}_{\mathcal{C}}(M,N)$ $\forall M,N \in \mathrm{ob}\,\mathcal{C}$

$$\begin{tikzcd} FM \arrow[r, "Ff"] \arrow[d, "\varphi_M"'] \arrow[dr, phantom, "\circlearrowright"] & FN \arrow[d, "\varphi_N"] \\ GM \arrow[r, "Gf"'] & GN \end{tikzcd}$$

$\varphi : F \to G$, $\psi : G \to H$ natural $\Rightarrow$ $\psi \circ \varphi : F \to H$ defined by $(\psi\circ\varphi)_X = \psi_X \circ \varphi_X$ is natural

A natural transformation $\varphi : F \to G$ is a natural isomorphism if $\exists \psi : G \longrightarrow F$ natural such that $\psi \circ \varphi = \mathrm{id}_F$ & $\varphi \circ \psi = \mathrm{id}_G$

Equivilently φ is a natural isomorphism $\Leftrightarrow$ $\forall X \in \mathrm{ob}\,\mathcal{C}$ φ_X is an isomorphism

Equivilences: $F : \mathcal{C} \longrightarrow \mathcal{D}$, a functor, is an equivilence of categories if $\exists G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms η_1, η_2 such that $F \circ G \overset{\eta_1}{\cong} \mathrm{id}_{\mathcal{D}}$ & $G \circ F \overset{\eta_2}{\cong} \mathrm{id}_{\mathcal{C}}$

If in addition $G \circ F = \mathrm{id}_{\mathcal{C}}$ & $F \circ G = \mathrm{id}_{\mathcal{D}}$ we have an isomorphism of categories.

The G here is a quasi-inverse to F & is not in general unique.

Functor Categories: For two categories $\mathcal{C}$ & $\mathcal{D}$ we define $\mathcal{C}^{\mathcal{D}}$ the category with objects functors $F : \mathcal{D} \longrightarrow \mathcal{C}$ and morphisms natural transformations

For a category $\mathcal{C}$ we define $\mathrm{PShv}(\mathcal{C})$ to be $\mathrm{Sets}^{\mathcal{C}^{op}}$.

Yoneda: The Yoneda functor is a functor $y : \mathcal{C} \longrightarrow \mathrm{PShv}(\mathcal{C})$

$X \longmapsto h^X : \mathcal{C}^{op} \longrightarrow \mathrm{Sets}$

$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$

$g \longmapsto (-) \circ g$

$(X \xrightarrow{f} Y) \longmapsto f \circ (-) : h^X \longrightarrow h^Y$

$\forall A \in \mathrm{ob}\,\mathcal{C}$ $(f \circ (-))_A : h^X A \longrightarrow h^Y A$

The Yoneda functor is fully faithful.

Abelian Categories:

R Modules:

In some sense every abelian category is a category over R-modules. So we first consider R-Mod defining properties.

Recall that a module over a ring R is an abelian group M with an action $R \times M \longrightarrow M$ satisfying vector space axioms.

The category of (left) R modules, denoted R-mod or Mod_R has

- objects: R modules
- morphisms: R-linear homomorphisms, that is group homomorphisms such that $\forall r \in R \ \forall m \in M \quad f(rm) = rf(m)$

For $f \in Hom_R(M,N)$. There is a cannonical (induced by universal property) map called the coimage hom.

$$coim(f) \longrightarrow im(f)$$

Where $coim(f) = coker(\ker(f) \hookrightarrow M)$
$\qquad = M/\ker(f)$

& recall that $im(f) = \ker(N \hookrightarrow coker(f))$

In the category Mod_R this map is an isomorphism $\forall f$.

Kernels:

The kernel of a morphism $f: X \longrightarrow Y$ in a more general setting is an object K & morphism $\varkappa : K \longrightarrow X$ such that

$$\begin{tikzcd} K \arrow[r, "\varkappa"] \arrow[dr, "0_{KY}"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

ie $f \circ \varkappa$ is the zero morphism from K to Y.

$$\begin{tikzcd} K \arrow[r, "\varkappa"] \arrow[dr, "0_{KY}"'] & X \arrow[d, "f"] & K' \arrow[l, "\varkappa'"'] \arrow[dl, "0_{K'Y}"] \arrow[ll, dashed, bend right, "\exists ! u"'] \\ & Y & \end{tikzcd}$$

i.e. given a morphism $\varkappa' : K' \to X$ such that $f \circ \varkappa' = 0_{K'Y}$ there is a unique morphism $u: K' \to K$ such that $\varkappa \circ u = \varkappa'$

The Hom sets of this category have a lot of structure:

- $f, g \in Hom_R(M,N)$ with operation $f+g$ defined by $(f+g)m = fm + gm$ form an abelian group
- $h \in Hom_R(N,Q)$ then $h \circ (f+g) = h \circ f + h \circ g$

(Also distributes from the left)

This is almost the structure of a ring.

Abelian Categories:

A category $\mathcal{C}$ with the natural structure of an abelian group (i.e composition on hom sets is bilinear) is called preadditive or Ab.

The dual concept is the cokernel. The kernel of a morphism is its coker in the oposite category.

Turn all the arrows around in the above diagram.

Given finitely many R-modules $M_1, \dots, M_n$ there is an R-module denoted $\bigoplus_{i=1}^{n} M_i$ with

- Elements: $(m_1, \dots, m_n)$ $(m_i \in M_i)$
- Component wise operations

This direct sum module is both a product & coproduct i.e. for an R-module T:

- $Hom_R(T, \oplus M_i) \xrightarrow{\sim} \prod Hom_R(T, M_i)$

Product $\quad f \longmapsto (\pi_1 \circ f, \dots, \pi_n \circ f)$ (projection)

- $Hom_R(\oplus M_i, T) \xrightarrow{\sim} \prod Hom_R(M_i, T)$

coproduct $\quad f \longmapsto (f \circ \iota_1, \dots, f \circ \iota_n)$ (inclusion)

A preadditive category with the further structure that

- There exists a zero object ie. $\exists\, 0 \in ob\mathcal{C} \quad \forall X \in ob\mathcal{C} \quad Hom(0,X) = Hom(X,0) = \{0\}$
- $\mathcal{C}$ admits all finite direct sums

is called an additive category.

The preadditive structure on an additive category is unique.

A functor, $F: \underline{A} \longrightarrow \underline{B}$, between two additive categories is called additive iff

- $F(0) = 0$
- $\forall X, Y \in ob\underline{A} \quad F(X \oplus Y) \cong FX \oplus FY$

For $f \in Hom_R(M,N)$ we define

- $\ker(f) = \{ m \in M : fm = 0 \}$

(This forms a submodule of M)

- $coker(f) = N / Im(f)$

Every homomorphism in mod_R admits a kernel & cokernel

An abelian category is an additive category such that:

- Each homomorphism admits a ker & coker
- All coimage homomorphisms are isomorphisms

$f: X \longrightarrow Y$ a morphism in some abelian category:

f isomorphism $\Leftrightarrow \underbrace{\ker(f)}_{inj} = \underbrace{coker(f)}_{surj}$

Adjoints & Limits:

Adjunctions:

A pair of functors $L: \underline{C} \rightleftarrows \underline{D}: R$, are an adjoint pair, (L,R) an adjunction iff $\exists$ a bijection τ such that

$\mathrm{Hom}_{\underline{D}}(LX, Y) \xrightarrow[\tau]{\sim} \mathrm{Hom}_{\underline{C}}(X, RY)$

and $\forall f \in \mathrm{Hom}_{\underline{C}}(X, X')$, $\forall g \in \mathrm{Hom}_{\underline{D}}(Y, Y')$)

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{D}}(LX', Y) \arrow[r, "-\circ Lf"] \arrow[d, "\tau"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\underline{D}}(LX, Y) \arrow[r, "g\circ -"] \arrow[d, "\tau"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\underline{D}}(LX, Y') \arrow[d, "\tau"] \\
\mathrm{Hom}_{\underline{C}}(X', RY) \arrow[r, "-\circ f"] & \mathrm{Hom}_{\underline{C}}(X, RY) \arrow[r, "Rg\circ -"] & \mathrm{Hom}_{\underline{C}}(X, RY')
\end{tikzcd}$$

For any adjoint pair $(L,R): \underline{C} \rightleftarrows \underline{D}$ there are natural transformations:

- Unit of adjunction: $\eta: id_{\underline{C}} \to R \circ L$
$$\begin{tikzcd}
X \arrow[r, "\tau f"] \arrow[d, "\eta_X"] \arrow[r, phantom, shift right=4, "\circlearrowleft"] & RY \\
RLX \arrow[ur, "Rf"'] &
\end{tikzcd}$$
- Counit of adjunction: $\varepsilon: LR \to id_{\underline{D}}$
$$\begin{tikzcd}
LX \arrow[r, "\tau^{-1} g"] \arrow[d, "Lg"] \arrow[r, phantom, shift right=4, "\circlearrowleft"] & Y \\
LRY \arrow[ur, "\varepsilon_Y"'] &
\end{tikzcd}$$

Further satisfying

★ $id_{LX} = \varepsilon_{LX} \circ L \circ \eta_X$; $id_{RY} = \eta_{RY} \circ R \circ \varepsilon_Y$

Given $(L, R, \eta, \varepsilon)$ satisfying ★ the maps $f \mapsto Rf \circ \eta$ makes L & R an adjoint pair.

We call an additive functor $F: \underline{A} \to \underline{B}$ between abelian categories:

- exact $\Leftrightarrow$ F preserves s.e.s $\Leftrightarrow$ F preserves BOTH co/kernels
- left exact $\Leftrightarrow$ F preserves kernels
- right exact $\Leftrightarrow$ F preserves cokernels

Now if $L: \underline{A} \rightleftarrows \underline{B}: R$ is an adjoint pair of additive functors then L is right exact & R is left exact.

Pointed Categories:

An initial object $c_\emptyset \in \underline{C}$ is an object such that $\forall d \in \underline{C}$ $\mathrm{Hom}_{\underline{C}}(c_\emptyset, d) = \{\varepsilon\}$ ← singleton.

Similarly a final object $c_f \in \underline{C}$ is an object st. $\forall d \in \underline{C}$ $\mathrm{Hom}_{\underline{C}}(d, c_f) = \{\varepsilon\}$

- $F: \emptyset \to \underline{C} \Rightarrow \mathrm{colim}\, F$ is initial in $\underline{C}$.
- $F: \emptyset \to \underline{C} \Rightarrow \lim F$ is final in $\underline{C}$

A category with an object that is BOTH initial & final is called pointed.

Co/Limits:

Let $\underline{A}, \underline{B}$ categories.

The colimit of a functor $F: \underline{A} \to \underline{B}$

$\mathrm{colim}_{\underline{A}}(F)$ is
- An object $\mathrm{colim}(F) \in \mathrm{ob}\,\underline{B}$
- Arrows $\forall a \in \underline{A}$ $F(a) \xrightarrow{\varphi_a} \mathrm{colim}(F)$ s.t. $\forall \gamma: a \to b$ $\varphi_a \circ \gamma = \varphi_b$ * and satisfy

$\forall b \in \underline{B}$ $\mathrm{Hom}_{\underline{B}}(\mathrm{colim}\, F, b) \hookrightarrow \prod_{a \in A} \mathrm{Hom}_{\underline{B}}(F(a), b)$

$f \longmapsto (f \circ \varphi_a)_{a \in A}$

This is injective with image $(g_a)_{a \in A}$ st. *.

$$\begin{tikzcd}
F(\underline{A}) \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=5, "\circlearrowleft"] & \mathrm{colim}(F) \arrow[d, dashed, "\exists!"] \\
& a \in \underline{A}
\end{tikzcd}$$

The limit of a functor, $\lim(F)$, is
- An object $\lim F \in \underline{B}$
- maps $F(\underline{A}) \longleftarrow \lim(F)$ such that

$\forall b \in \underline{B}$ $\mathrm{Hom}_{\underline{B}}(b, \lim F) \xrightarrow{\sim} \mathrm{Hom}_{\underline{B}}(b, F(\underline{A}))$

Co/limits are unique up to unique isomorphism.

Pushout & Pullback:

3 objects with 2 non-id morphisms.

Consider $\underline{D} = \begin{smallmatrix} d_1 & \to & d_2 \\ \downarrow & & \\ d_3 & & \end{smallmatrix}$ Then for any $\underline{C}$ a functor $F: \underline{D} \to \underline{C}$ is a diagram in $\underline{C}$. $\mathrm{Colim}(F)$ is called the pushout

$$\begin{tikzcd}
F(d_1) = C_1 \arrow[r] \arrow[d] & C_2 = F(d_2) \arrow[d, "1"] \arrow[ddr, bend left] & \\
F(d_3) = C_3 \arrow[r, "2"] \arrow[drr, bend right] & \mathrm{colim}(F) =: C_2 \amalg_{C_1} C_3 \arrow[dr, dashed, "\exists!\, \gamma"] & \\
& & \Omega
\end{tikzcd}$$

1 & 2 given by colimit universal property. Grey is the universal property of the pushout.

Note the morphisms are very important.

If γ is an iso then $\begin{smallmatrix} C_1 & \to & C_2 \\ \downarrow & & \downarrow \\ C_3 & \to & \Omega \end{smallmatrix}$ is cocartesian or a pushout square.

If $\underline{D} = h \to i \leftarrow j$ then $F: \underline{D} \to \underline{C}$

then $\lim(F) =: C_2 \times_{C_1} C_3$

$$\begin{tikzcd}
C_2 \times_{C_1} C_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_2 \arrow[d] \\
C_3 \arrow[r] & C_1
\end{tikzcd}$$

is the pullback with a similar universal property.

If the given arrow is an iso then $\begin{smallmatrix} \Omega & \to & C_2 \\ \downarrow & & \downarrow \\ C_3 & \to & C_1 \end{smallmatrix}$ is a cartesian square.

Co/Products:

A discrete category is one whose only morphisms are identities. Let $\underline{D}$ be such a discrete category & $F: \underline{D} \to \underline{C}$ a functor

- $\mathrm{Colim}(F) =: \coprod_{i \in D} F(i)$ called the coproduct & morphisms $\iota_i: F(i) \to \coprod_{i \in D} F(i)$ that satisfy the property; $\forall Y \in \underline{C}$ and any collection of morphisms $(f_j)_{j \in D}$ s.t. $f_k: F(k) \to Y$ $\exists! f$ such that $f_j = f \circ \iota_j$

$$\begin{tikzcd}
& F(i) \arrow[dl, "\iota"'] \arrow[dr, "f_i"] & \\
\coprod_{j \in D} F(j) \arrow[rr, dashed, "\exists!"] \arrow[rr, phantom, shift left=5, "\circlearrowleft"] & & Y
\end{tikzcd}$$

- $\lim(F) =: \prod_{i \in D} F(i)$ called the product.

with morphisms $\pi_j: \prod_{i \in D} F(i) \to F(j)$ satisfying property: $\forall Y \in \underline{C}$ and a family of morphisms $(f_i)_{i \in D}$ $\exists! f: Y \to \prod_{i \in D} F(i)$ such that $\forall i \in D$ this commutes

$$\begin{tikzcd}
\prod_{i \in D} F(i) \arrow[dr, "\pi_j"'] & & Y \arrow[ll, dashed, "\exists!"'] \arrow[dl, "f_j"] \\
& F(j) &
\end{tikzcd}$$

Cones:

The cone of a cochain map $f: C^\bullet \to D^\bullet$, denoted $\mathrm{cone}(f)$ is

- $\mathrm{Cone}(f)^n = C^{n+1} \oplus D^n$
- $d_{\mathrm{cone}(f)}(x, y) = (-d_C(x), d_D(y) - f(x)) = \begin{bmatrix} -d_C & 0 \\ -f & d_D \end{bmatrix}$

The cone is natural in f:

Given
$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "h"] \\
\tilde{C} \arrow[r, "\tilde{f}"] & \tilde{D}
\end{tikzcd}$$
a square of cochain complexes

We have a map $\varphi: \mathrm{cone} f \to \mathrm{cone} \tilde{f}$ that is compatible with the ses

$0 \to D \to \mathrm{Cone} f \to C[1] \to 0$

* Clarify THIS:

Given $f: C^\bullet \to D^\bullet$ chain map there is a natural long exact sequence

$\cdots \to H^n C \xrightarrow{f_*} H^n D \xrightarrow{inc} H^n \mathrm{cone} f \xrightarrow{proj} H^{n+1} C \to \cdots$

f is a qism $\Leftrightarrow$ $\mathrm{cone} f$ is acyclic.

Projectives & Injectives:

For $\mathcal{A}$ abelian $X \in \mathcal{A}$ h_X is left exact.

Also note that $h^X = h_{X^{op}}$.

$P \in \mathcal{A}$ is projective iff h_P is exact.

injective iff h^P is exact.

Necessary & Sufficient Conditions:

P projective $\Leftrightarrow \forall$
$$\begin{tikzcd} B \arrow[r, "\pi"] & B' \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\exists g"] & \end{tikzcd}$$
exact

E injective $\Leftrightarrow \forall$
$$\begin{tikzcd} 0 \arrow[r] & B' \arrow[r, "\iota"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists k"] \\ & E & \end{tikzcd}$$
exact.

For R-modules we have that P is projective $\Leftrightarrow \exists Q \in \underline{Mod}_R$ $P \oplus Q$ is free.

(Baers Criterion) $E \in \underline{Mod}_R$ is injective $\Leftrightarrow$ $\forall$ left ideals $J \subseteq R$ & homomorphisms $f: J \to E$ $\exists \tilde{f}: R \to E$.

Enough:

'$\mathcal{A}$ has enough projectives' iff $\forall X \in \mathcal{A}$ $\exists P \in \mathcal{A}$ projective and an epimorphism $P \xrightarrow{\pi} X$. "$X \cong P/\ker(\pi)$."

- '$\mathcal{A}$ has enough injectives' iff $\forall X \in \mathcal{A}$ $\exists E \in \mathcal{A}$ injective & a monomorphism $X \to E$

"Every object includes into another."

For any ring $\underline{Mod}_R$ has enough projectives and enough injectives.

For an aditive adjunction $L: \mathcal{A} \rightleftarrows \mathcal{B}: R$
- R exact $\Rightarrow$ L preserves projectives
- L exact $\Rightarrow$ R preserves injectives

Tensor Products:

Let k be a commutative ring.

Let $E_1, \dots, E_s \in \underline{Mod}_k$. An s-linear map $E_1 \times \cdots \times E_s \to M$ is a function k linear in each variable.

$M, N \in \underline{Mod}_k$ then the tensor product is:
- an object $M \otimes_k N \in \underline{Mod}_k$
- A bilinear map $T: M \times N \to M \otimes_k N$ satisfying
$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "T"'] & Q \\ M \otimes_k N \arrow[ur, dashed, "\exists! \tilde{\varphi}"'] & \end{tikzcd}$$

The tensor product corepresents $\mathrm{Bilin}_k(M \times N, -)$.

The following isomorphism are natural $\forall M, N, Q \in \underline{Mod}_k$
- $M \otimes_k N \cong N \otimes_k M$
- $(M \otimes_k N) \otimes_k Q \cong M \otimes_k (N \otimes_k Q)$
- $k \otimes_k M \cong M$.

For a fixed $M \in \underline{Mod}_k$ the functor
$$(- \otimes_k M): \underline{Mod}_k \longrightarrow \underline{Mod}_k$$
$$N \longmapsto N \otimes_k M$$
$$(N \xrightarrow{\varphi} Q) \longmapsto (N \otimes M \xrightarrow{\varphi \otimes id_M} P \otimes M)$$
$$(n \otimes m) \longmapsto \varphi(n) \otimes m$$
is left adjoint to $\mathrm{Hom}_k(M, -)$

moreover both are additive and $(- \otimes_k M)$ is right exact.

We say $M \in \underline{Mod}_k$ is flat iff $- \otimes_k M$ is exact.

Resolutions: $\mathcal{A}$ abelian

A projective resolution of $M \in \mathcal{A}$ is a sequence $P_\bullet$ of projective objects such that $H_s(P_\bullet) = \begin{cases} M, & s=0 \\ 0, & s>0 \end{cases}$

Injective resolution of M is a cochain complex of injective objects $I^\bullet$, with $H^s(I^\bullet) = \begin{cases} M, & s=0 \\ 0, & s>0 \end{cases}$.

If $\mathcal{A}$ has enough proj/inj then every $M \in \mathcal{A}$ admits a proj/inj resolution.

$f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$, $\overset{N \to I^\bullet}{}$ $P_\bullet \to M$ a $\overset{\text{inj-res}}{\text{proj-res}}$ & $\overset{M \to E^\bullet}{Q_\bullet \to N}$ a $\overset{\text{right}}{\text{left}}$ resolution $\Rightarrow$ $\exists \tilde{f}: \overset{E^\bullet \to I^\bullet}{P_\bullet \to Q_\bullet}$ unique up to chain homotopy such that $H_0(\tilde{f}) = f$.

(when $Q_\bullet \to N$ is also proj-res there is a chain homotopy equivalence $Q_\bullet \sim P_\bullet$.)

Horse Shoe Lemma: $0 \to A' \to A \to A'' \to 0$

a ses in $\mathcal{A}$ and $P'_\bullet \to A'$, $P''_\bullet \to A''$ proj-res' $\Rightarrow$ $\exists P_\bullet \to A$ proj-res (and maps) such that
$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$
is a split short exact sequence (in every degree).

Uniqueness of Resolutions:

- For $\mathcal{A}$ with enough injectives we can choose resolutions $I_A^\bullet$ for every $A \in \mathcal{A}$.
- Then we can define a functor ← left bounded
$$F: \mathcal{A} \longrightarrow K^+(\mathcal{A}) = Ch^+(\mathcal{A})/\text{homotopy}$$
by $A \longmapsto [I_A^\bullet]$ ← homotopy class
$$(A \xrightarrow{f} B) \longmapsto [I_A^\bullet \xrightarrow{\tilde{f}} I_B^\bullet]$$ ← lift from earlier lemma

(Note that $K^+(\mathcal{A})$ is not abelian).

- A different choice of resolutions leads to a uniquely naturally isomorphic functor.

Sequences & Chains:

We assume that $\mathcal{A}$ is a given abelian category and a full subcategory of mod_R for some R.

Exact Sequences:

Take $A, B, C \in ob\mathcal{A}$

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

The sequence of morphisms is exact at B iff $im(f) = ker(g)$. If its exact at all places it is called an exact sequence.

A short exact sequence (SES) is an exact sequence $0 \to A \to B \to C \to 0$

i.e.
- $A \to B$ is a monomorphism $ker = \{0\}$
- $B \to C$ is an epimorphism $coker = \{0\}$
- $ker(B \to C) = im(A \to B)$

Two sequences are isomorphic iff

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "\wr"] & X^n \arrow[r] \arrow[d, "\wr"] & X^{n+1} \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

there exist vertical isomorphisms such that this diagram commutes

A split exact sequence is any sequence that is isomorphic to the following ses

$$0 \to X \to X \oplus Y \to Y \to 0$$

$x \mapsto (x, 0)$

$(x, y) \mapsto y$

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

splits $\iff \exists s: Z \to Y \quad g \circ s = id_Z$

$\iff \exists r: Y \to X \quad r \circ f = id_X$

5 Lemma:

suppose the following commutes and has exact rows

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

f_1, f_2, f_4, f_5 are isomorphisms $\Rightarrow f_3$ is too.

Note:
- f_2, f_4 monomorphisms, f_1 epimorphism $\Rightarrow f_3$ monomorphism
- f_2, f_4 epimorphisms, f_5 monomorphism $\Rightarrow f_3$ epimorphism.

Chain Complexes:

A chain complex in $\mathcal{A}$ is a sequence of objects and morphisms

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

such that $\forall n \quad \partial_{n-1} \circ \partial_n = 0$.

The ∂_i are termed differentials or boundary maps.

A cochain complex has arrows in opposite direction & differentials denoted d^n.

A chain complex $(C_\bullet, \partial)$ is
- bounded below if $\exists N \ \forall n < N \quad C_n = 0$
- bounded above if $\exists N \ \forall n > N \quad C_n = 0$
- bounded if both

Category of Chains:

For $C^\bullet$ & $D^\bullet$ cochain complexes a chain map $C \xrightarrow{f} D$ is a sequence of maps $f^n : C^n \to D^n$ such that $d_D^n \circ f^n = f^{n+1} \circ d_C^n$

i.e. $\forall n$

$$\begin{tikzcd}
\cdots \arrow[r] & C^n \arrow[r, "d_C^n"] \arrow[d, "f^n"] \arrow[dr, phantom, "\circlearrowright"] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & D^n \arrow[r, "d_D^n"'] & D^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The cochain complexes in a given abelian category $\mathcal{A}$ form an abelian category $Ch(\mathcal{A})$ with objects chains & morphisms chain maps.

Chain Homotopies:

A chain homotopy of two chain maps $f, g : C_\bullet \to D_\bullet$ is a sequence of maps $s_n : C_n \to D_n$ st. $\forall n \quad f_n - g_n = d^D_{n+1} \circ s_n + s_{n-1} \circ d^C_n$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "f-g"] & C_n \arrow[r] \arrow[d, "f-g"] \arrow[dl, "s_n"] & C_{n-1} \arrow[r] \arrow[d, "f-g"] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We denote this $f \overset{s}{\sim} g$.

If $f \sim 0$ then f is null homotopic.

If $f : C_\bullet \to D_\bullet$ a chain map & $\exists g : D_\bullet \to C_\bullet$ st. $f \circ g \sim id_D$ & $g \circ f \sim id_C$ f is a chain homotopy equivalence.

$$f \sim g \Rightarrow f_* = g_*$$

Homology:

- $ker(\partial_n) = Z_n(C_\bullet)$ the module of n-cycles
- $Im(\partial_n) = B_{n-1}(C_\bullet)$ the module of n-1 boundries
- $H_n(C^\bullet) = Z_n(C_\bullet) / B_n(C_\bullet) = coker(C_{n+1} \xrightarrow{\partial} Z_n(C_\bullet))$

is the n^{th} homology of the chain $C_\bullet$.

$C_\bullet$ is acyclic $\iff \forall n \quad H_n(C_\bullet) = 0$

$\iff C_\bullet$ is exact

Similarly define the cohomology

$$H^n(C^\bullet) = ker(d^n) / Im(d^{n-1})$$

Alternatively we can consider

$$H^n : Ch(\mathcal{A}) \longrightarrow \mathcal{A} \quad \text{a functor}$$

$$C^\bullet \longmapsto H^n(A)$$

$$(C^\bullet \xrightarrow{\varphi} D^\bullet) \longmapsto (H^n(C^\bullet) \xrightarrow{\varphi^n_*} H^n(D^\bullet))$$

Note that $\varphi_*([c]) = [\varphi(c)]$.

A chain map f is a quasi-isomorphism (qism) $\iff \forall n \quad f^n_* : H^n C^\bullet \to H^n D^\bullet$ is an isomorphism.

Long Exact Sequence:

Consider a ses of cochain complexes

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

There are natural connecting homomorphisms (with respect to maps of ses') $\forall n \quad \delta = \delta^n : H^n C^\bullet \to H^{n+1} A^\bullet$ such that the following sequence is exact:

$$\cdots \to H^n A \xrightarrow{f_*} H^n B \xrightarrow{g_*} H^n C$$

$$\xrightarrow{\delta^n} H^{n+1} A \xrightarrow{f_*} H^{n+1} B \xrightarrow{g_*} H^{n+1} C$$

$$\xrightarrow{\delta^{n+1}} \cdots$$

The shifted cochain complex $C^\bullet[s]$ is defined by
- $C^\bullet[s]^n = C^{s+n}$
- $d^n_{C[s]} = (-1)^s d_C^{n+s}$

$$\cdots \to C^{n-1} \xrightarrow{d} C^n \to \cdots$$

$$\cdots \to C^n \xrightarrow{-d} C^{n+1} \to \cdots$$

Note that in homology we get $\forall n, s \quad H^n(C[s]) = H^{n+s}(C)$

In fact $S : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ is a functor (fix s)

$$C^\bullet \longmapsto C^\bullet[s]$$

$$f \longmapsto f[s] \qquad f[s]^n = f^{n+s}$$

Derived Functors:

Assumptions: $\underline{A}, \underline{B}$ abelian categories and all functors additive; (preserve chain complexes & chain homotopies).

We assume that if a category has enough inj/proj we assume that some resolution has been fixed for each object (choice doesn't matter by uniqueness discussion).

Derived Functors:

$F: \underline{A} \longrightarrow \underline{B}$ left exact, $\underline{A}$ has enough injectives: $R^s F: \underline{A} \longrightarrow \underline{B}$

$$A \longmapsto H^s F(I_A^\bullet)$$
$$f \longmapsto H^s F(\tilde{f})$$

$G: \underline{A} \longrightarrow \underline{B}$ left exact, $\underline{A}$ enough proj: $L^s G: \underline{A} \longrightarrow \underline{B}$

$$A \longmapsto H_s G(P_\bullet^A)$$
$$g \longmapsto H_s G(\tilde{g})$$

$F \xrightarrow{\sim} R^0 F$, $L_0 G \xrightarrow{\sim} G$

Universal δ Functors:

$F: \underline{A} \longrightarrow \underline{B}$ ~~right~~ left exact, $\underline{A}$ enough ~~proj~~ inj

$\Rightarrow \forall$ ses $0 \to A' \to A \to A'' \to 0$

$\exists\, \delta^s: L_s F(A'') \longrightarrow L_{s-1} F(A')$ $s+1$

connecting homomorphisms so

$$\cdots \xrightarrow{\delta_{s+1}} L_s F(A') \to L_s F(A) \to L_s F(A'') \xrightarrow{\delta^s} L_{s-1} F(A') \to L_{s-1} F(A) \to L_{s-1} F(A'') \xrightarrow{\delta^{s-1}} \cdots$$

is long exact.

The connecting homomorphisms are natural in the original ses.

Co/limits in Abelian Categories:

(AB3*) For every set of objects $\{A_i\}_{i \in I}$ the ~~coproduct~~ product exists.

(AB4*) AB3+ ~~coproduct~~ product of ~~monomorphisms~~ epimorphisms is a ~~monomorphism~~ epimorphism.

(AB5*) AB3 + ~~Filtered colimits~~ inverse limits are exact

No nonzero abelian category can satisfy both AB5 & AB5*.

Abelian category $\underline{A}$ is co/complete iff $\underline{A}$ satisfies AB3/AB3*

For $\underline{I}$ small $\underline{A}$ abelian $\underline{A}^{\underline{I}}$ is abelian.

We can write co/lim as functors

$$\text{co/lim}: \underline{A}^{\underline{I}} \longrightarrow \underline{A}$$
$$F \longmapsto \text{co/lim}_{\underline{I}}\, F$$

given by universal property

$$(F \xrightarrow{\varphi} G) \longmapsto \left(\text{co/lim}\, F \xrightarrow{\text{co/lim}\,\varphi} \text{co/lim}\, G\right)$$

Colim is left adjoint to $D: \underline{A} \longrightarrow \underline{A}^{\underline{I}}$ the constant diagram functor. $C \longmapsto \begin{pmatrix} i \mapsto C \\ (i \to j) \mapsto id_C \end{pmatrix}$

lim is right adjoint to something else.

(In particular they are right/left exact respectively).

$\underline{Mod}_R$ satisfies AB3, AB3*, AB4, AB4*, AB5

$F: \underline{Mod}_R \longrightarrow \underline{A}$ left adjoint functor, $\underline{A}$ satisfies AB4; let $\{M_i\}_{i \in I}$ be a set of objects in $\underline{mod}_R$ (indexed by a set I)

$\Rightarrow \forall s \geq 0$ $\bigoplus_{i \in I} L_s F(M_i) \cong L_s F(\bigoplus_{i \in I} M_i)$

k commutative ring, $M \in \underline{mod}_k$, $\underline{N}: \underline{I} \longrightarrow \underline{mod}_k$ a functor & $\underline{I}$ a filtered small category

$\Rightarrow \forall s \geq 0$ $\underline{\text{colim}}\, Tor_s^k(M, N_i) \cong Tor_s^k(M, \text{colim}\, N_i)$

Tor & Ext:

Tor: For a commutative ring R the left derived functor of $- \underset{R}{\otimes} M: \underline{Mod}_R \longrightarrow \underline{Mod}_R$ are $Tor_s^R(N, M) = L_s(- \underset{R}{\otimes} M)(N)$

Tor is balanced $Tor_s^R(N, M) \cong Tor_s^R(M, N)$

For A, B abelian groups & $s > 1$ $Tor_s^{\mathbb{Z}}(A, B) = 0$.

$Tor_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A_{tor}$

A is flat $\Leftrightarrow$ it is torsion free

For R a commutative ring, $M \in \underline{mod}_R$

M is flat $\Leftrightarrow$ $\forall$ ideals $I \trianglelefteq R$ $Tor_1^R(R/I, M) = 0$.

Ext: $\underline{A}$ abelian with enough injectives then the right derived functor of $Hom_{\underline{A}}(M, -)$ are the Ext functors.

$Ext_{\underline{A}}^s(M, N) = R^s Hom_{\underline{A}}(M, -)(N)$

If $\underline{A}$ has enough projectives then Ext is also balanced.

For $A, B \in \underline{A}$ an extension of A by B is a ses $0 \to B \to E \to A \to 0$

Two extensions are equivalent iff $\exists \varphi: E \to E'$ making

$$\begin{tikzcd}
 & E \arrow[d, "\varphi"] \arrow[dr] & \\
B \arrow[ur] \arrow[r] & E' \arrow[r] & A
\end{tikzcd}$$

Let ξ be an extension of A by B

We get a les in Ext

$$\cdots \to Hom(A, A) \xrightarrow[\delta]{} Ext^1(A, B) \to Ext^1(A, E) \to \cdots$$

Then we define $\Theta(\xi) = \delta(id_A)$

ξ splits $\Leftrightarrow \Theta(\xi) = 0$

$\left\{ \begin{matrix} \text{Equivalence classes of} \\ \text{extensions of } A \text{ by } B \end{matrix} \right\} \xleftrightarrow[\Theta]{\sim} Ext^1(A, B)$

Note that strictly Θ is a function on extensions but $\xi \sim \xi' \Rightarrow \Theta(\xi) = \Theta(\xi')$

Group (Co)Homology

Let G be a group, k a commutative group; The group ring kG is the free k-module on G $\bigoplus_{g\in G} k$, elements denoted $\sum_{g\in G} a_g g$ (allmost all $a_g=0$) with multiplication

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{h\in G} b_h h\Big) = \sum_{\gamma\in G}\sum_{\substack{g,h\\ \gamma=gh}} a_g b_h \gamma$$

A kG-module is a module over kG; equiv An abelian group M with a function $G\times M\to M$

- $e_G m = m$
- $\forall g,h\in G$ $(gh)m = g(hm)$
- $g(m+n)=gm+gn$
- $\forall \lambda\in k$ $g(\lambda m)=\lambda(gm)$

The functor

$\underline{\mathrm{mod}_{\mathbb{Z}G}} \longrightarrow \underline{Ab}$ is left exact.

$M \longmapsto M^G = \{m\in M : \forall g\in G\ gm=m\}$

The functor

$\underline{\mathrm{mod}_{\mathbb{Z}G}} \longrightarrow \underline{Ab}$ is right exact.

$M \longmapsto M_G = M/(m\sim m' \iff \exists g\ gm=m')$

Homology Functors

The cohomology of a group G, with coefficients in $M\in \underline{\mathrm{mod}_{\mathbb{Z}G}}$ is a sequence of abelian groups

$$H^n(G;M) = R^n(-)^G M$$
$$\cong \mathrm{Ext}^n_G(\mathbb{1}, M) = \mathrm{Ext}^n_G(\mathbb{Z}, M)$$

Homology is

$$H_n(G;M) = L_n(-)_G(M) \cong \mathrm{Tor}^G_n(\mathbb{Z}, M)$$

For a finite group G the norm element is $N=\sum_{g\in G} g \in \mathbb{Z}G$

- $N\in(\mathbb{Z}G)^G$
- $N^2 = |G|\cdot N$ card of G
- $(\mathbb{Z}G)^G = \mathbb{Z}N$

For a commutative ring k such that $s=|G|\hookrightarrow k$ is invertible we have

- $(N/s)^2 = N/s$
- $\forall x\in kG$ $(N/s)x = x(N/s)$
- For any $M\in \mathrm{mod}_{kG}$

$$H_0(G;M)\cong H^0(G;M)\cong \tfrac{N}{s}M$$
$$H_n(G;M)\cong H^n(G;M)=0 \quad n\geq 1.$$

Shapiros Lemma:

R a unital associative ring. M a right R-module, N a left R-module then

$$M\otimes_R N = M\otimes_{\mathbb{Z}} N \big/ \langle mr\otimes n - m\otimes rn\rangle$$

Note that $M\otimes_R N$ is not necessarily an R-module.

For $f:R\to S$ ring hom, M a left R-mod, N a left S-mod

$$\mathrm{Hom}_{\mathrm{mod}_S}(S\otimes_R M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{mod}_R}(M,N)$$
$$\varphi \longmapsto (m\mapsto \varphi(1\otimes_R m))$$

$H\leq G$, M a H-module then define

$$\left.\begin{aligned}\mathrm{Ind}_H^G(M) &= \mathbb{Z}G\otimes_{\mathbb{Z}H} M\\ \mathrm{CoInd}_H^G(M) &= \mathrm{Hom}_H(\mathbb{Z}G, M)\end{aligned}\right\}\ G\text{-modules} \quad (g\cdot\varphi)(x) = \varphi(g^{-1}x)$$

$\mathrm{Ind}_H^G:\underline{\mathrm{mod}_H}\longrightarrow\underline{\mathrm{mod}_G}$ is exact & left adjoint to restriction functor Res^H_G

$\mathrm{CoInd}_H^G:\underline{\mathrm{mod}_H}\longrightarrow\underline{\mathrm{mod}_G}$ is right adjoint to Res^H_G & is exact.

$H\leq G$. $M\in\underline{\mathrm{mod}_H}$

$$H_*(G,\mathrm{Ind}_H^G M)\cong H_*(H,M)$$
$$H^*(G,\mathrm{CoInd}_H^G M)\cong H^*(H,M)$$

$[G:H]<\infty \Rightarrow \exists \eta: \mathrm{Ind}_H^G \xrightarrow{\sim} \mathrm{CoInd}_H^G$

- G finite $\Rightarrow H^*(G,\mathbb{Z}G\otimes_{\mathbb{Z}} A)=0$, $\forall A\in \underline{Ab}$
- G finite P projective $\Rightarrow H^*(G,P)=H_*(G,P)=0$

Bar Resolution:

The unreduced Bar resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$ module is

$$\begin{tikzcd} \cdots \arrow[r] & B_n \arrow[r] & \cdots \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ & & & & \mathbb{Z} & \end{tikzcd}$$

B_n is the free $\mathbb{Z}G$ module on $G^{\times n}$ with elements denoted $[g_1,\dots,g_n]$, note $B^0=[\,]$.

The differential is $d: B_n\longrightarrow B_{n-1}$

$d=\sum_{i=0}^n (-1)^i d_i^n$ such that

$$\begin{cases} d_0[g_1,\dots,g_n] = g_1[g_2,\dots,g_n] \\ d_i[g_1,\dots,g_n] = [g_1,\dots,g_i g_{i+1},\dots,g_n] & 0<i<n \\ d_n[g_1,\dots,g_n] = [g_1,\dots,g_{n-1}] \end{cases}$$

Group Extensions

A group extension of G by abelian group A is $0\longrightarrow G\xrightarrow{\iota} E\xrightarrow{\pi} A\longrightarrow 1$ a ses. i.e. $G\cong E/A$

The semi-direct product of groups A & G by $\varphi:G\longrightarrow \mathrm{Aut}(A)$ is $A\rtimes_\varphi G$

- Set $A\times G$
- Multiplication: $(a,b)\cdot(c,d) = (a\varphi(b)(c), bd)$

An extension is split iff $\exists\sigma: G\to E$, $\pi\circ\sigma = id_G$ iff there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\wr"] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A \arrow[r] & A\rtimes G \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

Let $0\to A\to E_i\xrightarrow{\pi_i} G\to 1$, $i=1,2$ be two extensions such that there are set functions $s_i: G\longrightarrow E_i$, $\pi_i\circ s_i = id_G$, the brackets $[,]_1=[,]_2$ and $G\curvearrowright_{E_1} A = G\curvearrowright_{E_2} A$ then $E_1\cong E_2$.

PROOFS:

The Yoneda functor is fully faithful.

$Y : \underline{C} \longrightarrow PShv(\underline{C})$

We need to show that for any $X, Z \in C$
$Y_{X,Z} : Hom_C(X,Z) \longrightarrow Hom_{PShv(\underline{C})}(YX, YZ)$ is bijective.

① Faithful: Injectivity of Y_{XZ}.
Let $f, g : X \to Z$ and assume
$Y_{X,Z}(f) = h^f = h^g = Y_{X,Z}(g)$.

Consider $id_X \in h^X(X) = Hom_{\underline{C}}(X,X)$

Because the natural transformation $h^f : h^X \to h^Z$ sends morphisms to precompositions →

Then $f = f \circ id_X = h^f(id_X)$
$= h^g(id_X)$
$= g \circ id_X = g$ □

② Full: Surjectivity of Y_{XZ}.
Let $\varphi : h^X \longrightarrow h^Z$ a natural transformation.
Let $f = \varphi_X(id_X)$

$$\begin{array}{ccc} h^X(X) & & id_X \\ \varphi_X \downarrow & & \downarrow \\ h^Z(X) & & \varphi(id_X) \\ \parallel & & \\ Hom_{\underline{C}}(X,Z). & & \end{array}$$

Claim: $\varphi = h^f$.
$f = \varphi_X(id_X) : X \longrightarrow Z$.

Check equality of all φ_W.
{ Next let $W \in \underline{C}$,
& $\alpha \in h^X(W) = Hom_{\underline{C}}(W,X)$

Then $h^f(\alpha) = f \circ \alpha$
$= \varphi_X(id_X) \circ \alpha$
$= \varphi_W(id_X \circ \alpha)$ } Naturality.
$= \varphi_W(\alpha)$

Draw the diagram.

In $\underline{\text{Mod}}_R$
$\mathrm{coim}(f) \xrightarrow{\sim} \mathrm{im}(f)$
for every f

Consider $c : \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$
for $f \in \mathrm{Hom}_{\underline{\text{mod}}_R}(M, N)$

c is induced by universal property where

$$\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \hookrightarrow M) \xrightarrow{c} \mathrm{Im}(f) = \ker(N \hookrightarrow \mathrm{coker}(f)).$$

$$\underbrace{m + \ker(f)}_{\text{Additive coset}} \longmapsto f(m)$$

↑ induced by universal property of the cokernel & kernel

- Induced morphism
- Surjective clear
- Injective: $\ker(c) = \{ m + \ker(f) : \underline{f(m) = 0} \} = \{ 0 + \ker(f) \}$

  $\hookrightarrow m \in \ker(f) \Rightarrow [m] = [0]$ under $M/\ker(f)$.

  $\Rightarrow c$ injective.

Note that induced maps are unique so we also need to check that the map as defined satisfies the universal property.

$\boxed{\text{5 Lemma}}$ (In an abelian category $\mathcal{A}$)
Consider the diagram with exact rows

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "G"] & B \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "G"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & D \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "G"] & E \arrow[d, "\wr"] \\
\alpha \arrow[r] & \beta \arrow[r] & \kappa \arrow[r] & \delta \arrow[r] & \varepsilon
\end{tikzcd}$$

$C \xrightarrow{f} \kappa$ is isomorphism $\iff$ $\operatorname{coker}(f) = \ker(f) = 0$

$\mathcal{A}$ abelian $\Longrightarrow$ $\mathcal{A}^{op}$ abelian
We also have that $\ker(f) = \operatorname{coker}(f^{op})$
But the other maps will still be isomorphisms in the opposite category so we could apply the result of $\ker(f)$ being zero.

i.e. It suffices to prove $\ker(f) = 0$.

Proof:

$$\begin{tikzcd}
A \arrow[r, "\omega_1"] \arrow[d, "f_1", "\wr"'] \arrow[dr, phantom, "G"] & B \arrow[r, "\omega_2"] \arrow[d, "f_2", "\wr"'] \arrow[dr, phantom, "G"] & C \arrow[r, "\omega_3"] \arrow[d, "f"] \arrow[dr, phantom, "G"] & D \arrow[r] \arrow[d, "\wr", "f_3"'] \arrow[dr, phantom, "G"] & E \arrow[d, "\wr"] \\
\alpha \arrow[r, "g_1"'] & \beta \arrow[r, "g_2"'] & \kappa \arrow[r, "g_3"'] & \delta \arrow[r] & \varepsilon
\end{tikzcd}$$

let $x \in \ker(f)$ $\Longrightarrow$ $g_3(f(x)) = 0 = f_3(\omega_3(x)) = 0$
(Commutes)

$\Longrightarrow$ $\omega_3(x) = 0$ (f_3 iso)

$\Longrightarrow$ $\exists b \in B$ $\omega_2(b) = x$ ($\ker \omega_3 = \operatorname{Im} \omega_2$) Exactness.

$\Longrightarrow$ $f(\omega_2(b)) = g_2(f_2(b)) = 0$ (Commutes)

$\Longrightarrow$ $f_2(b) \in \ker(g_2) = \operatorname{Im} \alpha$

$\Longrightarrow$ $\exists a \in \alpha$ $g_1(a) = f_2(b)$

$\Longrightarrow$ $\exists a' \in A$ $g_1(f_1(a')) = f_2(b)$

$\Longrightarrow$ $f_2(\omega_1(a') - b) = f_2(\omega_1(a')) - f_2(b)$
$= f_2(b) - f_2(b)$
$= 0$

$\Longrightarrow$ $\omega_1(a') - b = 0$ (f_2 iso)

$\Longrightarrow$ $\omega_2(\omega_1(a')) = x = 0$ (chain)

$\square$

Long Exact Sequence

Consider a ses of cochain complexes

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

There are natural maps $\delta^s : H^s C^\bullet \longrightarrow H^{s+1} A \quad \forall s$
such that $\cdots \to H^n A \xrightarrow[f_*]{} H^n B \xrightarrow[g_*]{} H^n C \xrightarrow[\delta_n]{} H^{n+1} A \longrightarrow \cdots$ is les.

Proof: ① Create δ:

② Show exactness (not done in class).

$\boxed{f \sim g \Rightarrow f_* = g_*}$

Let $f \sim g : C_\bullet \longrightarrow D_\bullet$ chain maps

Now $f_* = g_* \iff (f_* - g_*) = 0$
$\iff (f - g)_* = 0$

So we will show only that $f \sim 0$
$\Longrightarrow f_* = 0$.

Proof: Let $[c] \in H_n C_\bullet$

$$\begin{aligned}
\Rightarrow f_*([c]) &= [f(c)] \quad (\text{Def of } f_*) \\
&= [(s \circ d_n)(c) + (d_{n+1} \circ s)(c)] \quad (d \,\&\, s \text{ differential } \& \text{ chain map of } f \text{ to } 0.) \\
&= [s(0) + (d \circ s)(c)] \\
&= [(d \circ s)(c)] = 0
\end{aligned}$$

$[c] \in H_n C_\bullet = \ker(d_n) / \operatorname{Im}(d_{n+1})$
$\Rightarrow c \in \ker(d_n)$

$(d \circ s)(c) \in \operatorname{Im}(d_{n+1})$

$\Rightarrow [(d \circ s)(c)] = 0 \in \ker(d_n) / \operatorname{Im}(d_{n+1})$

The cone is neutural in Chain maps.

We prove that

$$\begin{tikzcd}
C^\bullet \arrow[r, dashed] \arrow[rr, bend right, "f"'] \arrow[d, "g"'] & \operatorname{cone} f \arrow[r, dashed] \arrow[d, "G"] & D^\bullet \arrow[d, "h"] \\
\bar{C}^\bullet \arrow[r, dashed] \arrow[rr, bend right, "\bar{f}"'] & \operatorname{cone} \bar{f} \arrow[r, dashed] & \bar{D}^\bullet
\end{tikzcd}$$

given the black square the cone can be fit into the diagram such that the black square commutes.

Define $\operatorname{cone}(f) \xrightarrow{\varphi} \operatorname{cone}(\bar{f})$ by

$$\varphi^n = \begin{bmatrix} g^{n+1} & 0 \\ 0 & h^n \end{bmatrix}$$

Just checks this is a chain map but also need commutativity.

A chain map $f: C_\bullet \longrightarrow D_\bullet$ is a quasi-iso $\Leftrightarrow$ $\operatorname{Cone}(f)$ is acyclic.

Take cone les

$$\to H^{n-1}\operatorname{cone} f \xrightarrow{\pi_*} H^n C \xrightarrow{f_*} H^n D \xrightarrow{\iota_*} H^n \operatorname{Cone} f \to \cdots$$

f_* is iso $\Leftrightarrow$ $\ker(f_*) = \operatorname{coker}(f_*) = 0$

$\Leftrightarrow$ $\pi_*^n = \iota_*^n = 0$. $\forall n$

$\Leftrightarrow$ $H^n \operatorname{Cone} f = 0$

$$H^n D \xrightarrow{\iota_*} H^n \operatorname{cone} f \xrightarrow{\pi_*} H^{n+1} C$$

$\Leftrightarrow$ cone f acyclic $\square$

Another reference to naturality where it doesn't belong.

Let $(L,R): \underline{C} \longrightarrow D$ be an adjoint pair.

$$\begin{tikzcd} X \arrow[r, "\tau f"] \arrow[d, "\eta_x"'] \arrow[dr, phantom, "\circlearrowleft"] & RY \\ RLX \arrow[ur, "Rf"'] & \end{tikzcd}$$

Recal $\tau: Hom(LX,Y) \xrightarrow{\sim} Hom(X,RY)$ natural.

$Y = LX$,

We let $\eta_x : X \longrightarrow RLX$, $x \longmapsto \tau(id_{LX})(x)$

Then check $\tau f = \tau(f \circ id_{LY})$
$= Rf \circ \tau(id_{LX})$ (naturality of τ)
$= Rf \circ \eta_x$

Still need to check η defined like this is a natural transformation $id \longrightarrow R \circ L$.

$\eta : id_{\underline{C}} \longrightarrow RL$ natural $\Leftrightarrow$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_x"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\eta_y"] \\ RLX \arrow[r, "RLf"'] & RLY \end{tikzcd}$$

$$\begin{aligned} \eta_y f &= \tau(id_{LY})(f) \\ &= (- \circ f)\tau(id_{LY}) \\ &= \tau(id_{LY} \circ Lf) \\ &= \tau(Lf) \\ &\overset{?}{\vdots} \\ &= RLf(\eta_x) \end{aligned}$$

Think using

$LX \longrightarrow LRLX \longrightarrow LX = id$

but not from τ diagram (id doesn't fit).

$(L, R): \underline{A} \longrightarrow \underline{B}$
an adjoint pair of additive functors.
$\Rightarrow L$ is right exact & R is left exact

By symmetry we only show right exactness of L.

So let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$ be a short exact sequence in $\underline{A}$

Right exactness is preserving cokernels?

Right exactness means preserving cokernels. So we want to show that $LA' \xrightarrow{Li} LA \xrightarrow{Lp} LA'' \longrightarrow 0$ is exact.

What does this mean.
↳ Is there a statement of ses in terms of ker & coker agreeing?

$Lp \circ Li = L(p \circ i) = L(0) = 0$ by additivity. This shows that above is a chain.

Next need that Lp epimorphism $\}$
and $\ker(Lp) = \operatorname{Im}(Li)$. $\}$

Didnt show in class?

showed that Lp is coker of Li.
Why is this equivalent.

$$\begin{array}{c} \text{coker} \\ \| \end{array}$$

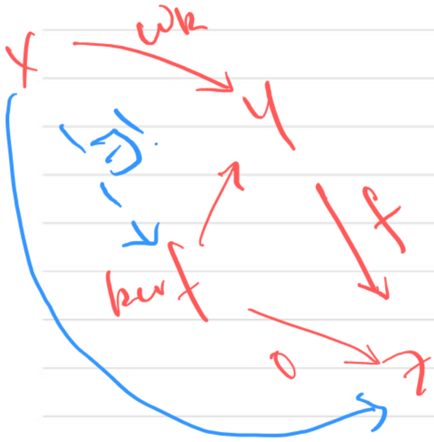
$$A \xrightarrow{f} B \xrightarrow{\text{coker}} C \longrightarrow 0.$$

$$C \cong B/\operatorname{Im} f$$

$\{$ ker = im & surj

$\boxed{h_X \text{ is left exact}}$

For $X \in \mathcal{A}$ an abelian category.



No idea there.

We show that it preserves kernels. i.e. let $f: Y \to Z$ given.

Then it has kernel
$\ker f \xrightarrow{\iota} Y$

We want to show under that

$$h_X(\ker f \xrightarrow{\iota} Y) = \ker(h_X f) \xrightarrow{\iota_X} h_X Y .$$

$$\Leftrightarrow \mathrm{Hom}(X, \ker f) \xrightarrow{\iota_*} \mathrm{Hom}(X, Y)$$

$$= \ker(f_*) \xrightarrow{\iota_*} \mathrm{Hom}(X, Y)$$

$$\Leftrightarrow \mathrm{Hom}(X, \ker f) = \ker(f_*)$$

$\Leftrightarrow \mathrm{Hom}(X, \ker f)$ satisfies universal property of kernel i.e.

$$\begin{tikzcd}
\mathrm{Hom}(X, \ker f) \arrow[r] \arrow[dr, "0"'] & \mathrm{Hom}(X, Y) \arrow[d, "f_* = f \circ -"] & K \arrow[l, "\omega"'] \arrow[dl, "0"] \arrow[ll, bend right, dashed, "?"'] \\
& \mathrm{Hom}(X, Z) &
\end{tikzcd}$$

$$f(\omega(k)) = 0 : X \to Z \text{ map}$$

i.e. $\omega(k) \in \ker(f)$

$P \in \underline{\mathrm{Mod}}_R$ is projective
$\iff$
$\exists Q \in \underline{\mathrm{Mod}}_R$ st $P \oplus Q$ is free

($\Leftarrow$) Let Q be given such that $P \oplus Q$ is free.

Then consider

$$\begin{tikzcd}
P \oplus Q \arrow[r, shift left, "\pi"] \arrow[d, dashed, "\exists"'] & P \arrow[l, shift left, bend left, "\iota"] \arrow[d, "f"] \arrow[dl, dashed] & \\
B \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$

$P \oplus Q$ is free so in particular projective

By composing the inclusion & map given by $P \oplus Q$ we get projectivity of P.

($\Rightarrow$) Let P be projective.

Then consider

Where F is the free module generated by P, i.e.
$F = R\langle u(P) \rangle$.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists"'] \arrow[d, equal] & \\
F \arrow[r, "\pi"] & P \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

The map given by projectivity is a section $P \longrightarrow F$ thus

$$F \cong P \oplus \ker(\pi)$$

$\square$

$E \in \mathrm{mod}_R$ is injective $\iff$ $\forall$ left ideals $J \subseteq R$ and homomorphisms $J \to E$ there is an extension $R \to E$.

injective iff a hom on ideal can be extended to whole ring.

($\Rightarrow$)
$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "\iota"] \arrow[d] & R \arrow[dl, dashed, "\exists"] \\ & E & \end{tikzcd}$$
Immediate from def of injective object.

($\Leftarrow$) An argument using Zorns Lemma.
A poset where every chain has a bound contains a maximal element.

We prove given
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & N \\ & E & \end{tikzcd}$$
There is an extension of f to $N \to E$.

So let $f : M \to E$ be given

Define $\mathcal{S} = \{ (W, g : W \to E) : M \subseteq W \subseteq N,\ g|_M = f \}$

Clearly $\mathcal{S}$ is nonempty because by assumption $(M, f) \in \mathcal{S}$
It is also partially ordered by inclusion.

Let $(W_i, g_i)_{i \in I}$ be a chain in $\mathcal{S}$.
There is an upper bound, namely $(\bigcup_{i \in I} W_i, \bigcup_{i \in I} f_i) \in \mathcal{S}$
in $\mathcal{S}$ b/c :
- Union of submodules is a submodule and
- $(\bigcup f_i)(x) = f_j(x)$ $x \in M_j$ a well defined map.

Now by Zorns Lemma $\exists (\omega, \tilde{f}) \in \mathcal{S}$ maximal.
i.e. $(\omega, \tilde{f}) \subseteq (\omega', \tilde{f}') \Rightarrow \omega = \omega'$.

Now we show that $\omega = N$ (thus completing the proof):
Assume for a contradiction $\omega \neq N \Rightarrow \exists x \in N - \omega$
Let $J = \{ r \in R : rx \in \omega \}$. J is clearly an ideal of R.

Define $g : J \to E$, $r \mapsto \tilde{f}(rx)$ a mod_R homomorphism.
By hypothesis we can extend
$$\tilde{g} : R \to E,\ \tilde{g}|_J = g.$$
Then let $Q = \omega + Rx \subseteq N$, which we have assumed $\omega \subsetneq Q$. But then we have a hom extending $\tilde{f}$
$\varphi : Q \to E$, $m + rx \mapsto \tilde{f}(m) + \tilde{g}(r)$ contradicting maximality of ω. $\otimes$

Thus $\omega = N$ & we have extended an arbitrary f $\square$.

$L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ an additive adjunction

R exact $\Rightarrow L$ preserves projectives

Let R be exact, $P \in \mathcal{A}$ projective and $B \to B' \to 0$ exact such

By projectivity of P & R exact then

$$\begin{tikzcd} RB \arrow[r] & RB' \arrow[r] & 0 \\ & P \arrow[ul, dashed, "\exists g"] \arrow[u, "f"'] & \end{tikzcd}$$

$\tau^{-1}(g) : LP \to B$ thus we have constructed

$$\begin{tikzcd} B \arrow[r] & B' \arrow[r] & 0 \\ & LP \arrow[ul, dashed, "\tau^{-1}(g)"] \arrow[u, "\tau^{-1}(f)"'] & \end{tikzcd}$$

$\square$

The tensor product of two modules always exists

(modules over a commutative ring)

Let $M, N \in \underline{\text{Mod}}_k$

Let $L(M \times N)$ be the free k-module on $M \times N$. i.e. taking $M \times N$ as a set of generators.

There is a set function

$$M \times N \longrightarrow L(M \times N)$$
$$m, n \longmapsto 1 \cdot (m, n)$$

$1 \in k$ generator module operation

Let $R \subseteq L(M \times N)$ be the submodule generated by

$$\{1 \cdot (rm_1 + sm_2, n) - r(m_1, n) - s(m_2, n) : r, s \in k, m_i \in M, n \in N\}$$
$$\cup \{1 \cdot (m, rn_1 + sn_2) - r(m, n_1) - s(m, n_2) : r, s \in k, m_i \in M, n \in N\}.$$

Designed to be such that map is bilinear.

$$M \times N \longrightarrow L(M \times N)/R$$
$$m, n \longmapsto 1 \cdot (m, n) + R = m \otimes n$$

is the tensor product (satisfies the universal property).

$$\boxed{\begin{aligned} M\otimes N &\cong N\otimes M \\ (M\otimes N)\otimes Q &\cong M\otimes(N\otimes Q) \\ k\otimes M &\cong M \end{aligned}}$$

Recall the universal property of tensor

$$\begin{tikzcd} M\times N \arrow[r, "\varphi"] \arrow[d, "T"', "\text{Bilinear}"] & Q \\ M\otimes N \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

i) Notice that

$$\begin{tikzcd} N\times M \arrow[r, "\varphi"] \arrow[d, "f"] & K \\ M\times N \arrow[ur, "\varphi\circ f^{-1}"] \arrow[d, "T"] & \\ M\otimes N \arrow[uur, dashed] & \end{tikzcd}$$

By universal property.

ii) Trilinear maps

iii)

$N \longmapsto N \otimes_k M$ is left adjoint to $\mathrm{Hom}_k(M,-)$

We need to show
$\exists J : \mathrm{Hom}_{\mathrm{mod}_k}(X \otimes_k M, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{mod}_k}(X, \mathrm{Hom}_{\mathrm{mod}_k}(M, Y))$
that satisfies naturality conditions.

Note that we are in an abelian category so Hom sets have abelian structure, moreover the commutativity of k gives them module structure.

$$\mathrm{Hom}_k(M \otimes_k M, P) \cong \mathrm{Bilin}_k(M \times N, P) \xrightarrow{\sim} \mathrm{Hom}_k(M, \mathrm{Hom}_k(N, P))$$

Tensor universal property

$$\varphi \longmapsto \psi$$
$$m,n \mapsto p \qquad\qquad m \longmapsto \varphi(m,-)$$

$$\psi \longleftarrow \psi$$
$$(m,n) \longmapsto \psi(m)(n)$$

• Show this makes the relevant diagram commute

If $\mathcal{A}$ has enough projectives then every $M \in \mathcal{A}$ admits a projective resolution

p', p Always can add an epi from a projective but it wouldn't be exact (so not a resolution).

$$\begin{tikzcd}
P_1 \arrow[rr, "\iota_0 \circ \pi_1"] \arrow[dr, "\pi_1"'] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& \ker \varepsilon \arrow[ur, "\iota_0"'] & & &
\end{tikzcd}$$

enough

Is an exact section by construction.
& we continue in this fashion indefinitely.

$f: M \to N$ hom; $P_\cdot \to M$ a projective res; $Q_\cdot \to N$ left res; $\Longrightarrow$ $\exists \tilde{f}: P_\cdot \to Q_\cdot$ s.t.

$$\left.\begin{tikzcd} P_0 \arrow[r, "\tilde{f}_0"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Q_0 \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd}\right\}$$ i.e $H_0(\tilde{f}) = f$

& $\tilde{f}$ is unique up to homotopy.

Construct $\tilde{f}$: First we denote the kernel of $Q_i \longrightarrow Q_{i-1}$ by $K_i \longrightarrow Q_i$. Then consider

P_0 projective.

$$\begin{tikzcd} P_3 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\exists\ \tilde{f}_0"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q_3 \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & \underbrace{Q_0 \arrow[r] & N}_{\text{epic}} \arrow[r] & 0 \end{tikzcd}$$

Inductive step: Now given $\tilde{f}_0, \dots, \tilde{f}_{n-1}$ such that $\tilde{f}_i \circ d_P = d_Q \circ \tilde{f}_{i+1}$ $(i < n-1)$

Then $d_Q \circ \tilde{f}_{n-1} \circ d_P = f_{n-2} d_P d_P = 0$

$\Rightarrow$ $\tilde{f}_{n-1} d_P$ factors through $K_{n-1} = \ker(d_{Q_{n-1}})$

$\operatorname{Im}(f_{n-1} \circ d_P) \subseteq \ker(d_{Q_{n-1}})$

By projectivity

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, dashed, "\exists \tilde{f}_n"'] \arrow[dr, "\tilde{f}_{n-1} d_P"] & P_{n-1} \arrow[r] \arrow[d, "\tilde{f}_{n-1}"] & P_{n-2} \arrow[r] \arrow[d, "\tilde{f}_{n-2}"] & \cdots \\ Q_n \arrow[r] \arrow[r, "\text{epic b/c acyclic}"'] \arrow[ur, phantom] & Q_{n-1} \arrow[r] & Q_{n-2} \arrow[r] & \cdots \end{tikzcd}$$

$$Q_n \to K_{n-1} \to Q_{n-1}$$

factors through

universal prop of ker.

So by induction we are are done.

$$\begin{tikzcd} P_\bullet \arrow[r, "\tilde F - \tilde f"] \arrow[d] \arrow[dr, phantom, "G"] & Q_\bullet \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

Next we show $\tilde f$ is unique up to Homotopy:

Let $\tilde F$ cover f too.
Then $H_0(\tilde F - \tilde f) = 0$
So WLOG we show $\tilde f \sim 0$ when $f = 0$.
i.e uniqueness of $\tilde f$ up to homotopy

$$\begin{tikzcd} P_\bullet \arrow[r] \arrow[d, "\tilde f"] \arrow[dr] & 0 \\ Q_\bullet \arrow[r] \arrow[dr] & 0 \\ & N \end{tikzcd}$$
M, f

So let $f = 0$ & $\tilde f$ be given. $H_* P_\bullet \xrightarrow{\tilde f_*} H_* Q_\bullet$
By hypothesis $H_0(\tilde f) = f = 0$ $\{ \tilde f_0[c] = [\tilde f c] = 0$

$\Rightarrow \tilde f_0$ factors through $E_0 = \operatorname{im}(d_Q) = \ker(\varepsilon)$

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r] \arrow[d, "\tilde f_3"] & P_2 \arrow[r] \arrow[d, "\tilde f_2"] & P_1 \arrow[rr] \arrow[d, "\tilde f_1"] & & P_0 \arrow[r] \arrow[d, "\tilde f_0"] \arrow[dl, "\tilde f_0"'] \arrow[dlll, bend right, "s_0"'] & 0 \arrow[dl, "s_{-1}"] \\
\cdots \arrow[r] & Q_3 \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[rr, "d_Q"] \arrow[r] & E_0 \arrow[r] & Q_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
& & & & & N \arrow[ur, dashed] &
\end{tikzcd}$$
G

Then we get s_0 by projectivity of P_0

$\Rightarrow \tilde f_0 = d_Q \circ s_0 + 0 = d_Q \circ s_0 + s_{-1} \circ d_P$

Next inductive step: Consider
$$\begin{aligned} d_Q \circ (\tilde f_1 - s_0 \circ d_P) &= d_Q \tilde f_1 - d_Q s_0 d_P \\ &= d_Q \tilde f_1 - \tilde f_0 d_P \\ &= 0. \end{aligned}$$

$\Rightarrow \tilde f_1 - s_0 d_P$ factors through kernel of d_Q

Then we repeat the above construction.

Let $P_\bullet \to M$ & $Q_\bullet \to M$ projective Resolutions
$\Rightarrow \exists$ a chain homotopy equivilence
$P_\bullet \longrightarrow Q_\bullet$ covering id_M, moreover it is unique up to homotopy.

Let $f: M \longrightarrow M$ be id_M then by previous thm $\exists \alpha, \beta$ chain maps unique up to homotopy

$\alpha: P_\bullet \longrightarrow Q_\bullet : \beta$ covering id_M.

$\Rightarrow \alpha \circ \beta$ covers the id_M & $\beta \circ \alpha$ too

$\Rightarrow \alpha \circ \beta \sim id_{P_\bullet}$, $\beta \circ \alpha \sim id_{Q_\bullet}$ (by uniqueness)

$\Rightarrow \alpha$ & β are mutually inverse. (i.e. chain homotopy equivilence).

Horse Shoe Lemma: $0 \to A' \to A \to A'' \to 0$ ses in $\underline{A}$

$P_\bullet' \to A'$, $P_\bullet'' \to A''$ projective resolutions.

$\Rightarrow$ There is a ses of split projective resolutions of $0 \to A' \to A \to A'' \to 0$ given by $0 \to P_\bullet' \to P_\bullet \to P_\bullet'' \to 0$.

Let $P_i = P_i' \oplus P_i''$. The maps are inclusion & projection. Then we proceed to construct the differentials of the resolution.

So we have

$$\begin{tikzcd}
P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] & A' \arrow[d, "i"] \\
P_0 = P_0' \oplus P_0'' \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & A \arrow[d, two heads, "p"] \\
P_0'' \arrow[r] \arrow[ur, "\gamma"] & A''
\end{tikzcd}$$

p is epic so by projectivity of P_0'' we get γ.

So let $\varepsilon = \underbrace{i \circ \varepsilon'}_{\text{first compont}} + \underbrace{\gamma}_{\text{second compont}}$

The two diagrams clearly commute by construction (or calculation)

ε is epimorphism b/c $A'' \cong A/A'$ and ε' is surjective onto A', while ε'' is surjective onto A'' hence ε is surjective onto A.

Then use the familiar construction:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & \\
P_n' \arrow[r, two heads, "d'"] \arrow[d, "\iota"] & \ker(d_{n-1}') \arrow[r, tail] \arrow[d] & P_{n-1}' \arrow[r, "d_{n-1}'"] \arrow[d, "\iota"] & \cdots \\
P_n = P_n' \oplus P_n'' & \ker(d_{n-1}) \arrow[r, tail] \arrow[d] & P_{n-1} = P_{n-1}' \oplus P_{n-1}'' \arrow[r, "d_{n-1}"] \arrow[d, "\pi"] & \cdots \\
P_n'' \arrow[r, two heads, "d''"] \arrow[d] \arrow[u, leftarrow, "\pi"] & \ker(d_{n-1}'') \arrow[r, tail] & P_{n-1}'' \arrow[r, "d_{n-1}''"] \arrow[d] & \cdots \\
0 & & 0 &
\end{tikzcd}$$

Commutativity of constructed sections

exactness

Consider the ses of chains:

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_{n-1}') \arrow[r] \arrow[d] & P_{n-1}' \arrow[r] \arrow[d] & \ker(d_{n-2}') \arrow[r] & 0 \\
0 \arrow[r] & \ker(d_{n-1}) \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \ker(d_{n-2}) \arrow[r] & 0 \\
0 \arrow[r] & \ker(d_{n-1}'') \arrow[r] & P_{n-1}'' \arrow[r] & \ker(d_{n-2}'') \arrow[r] & 0
\end{tikzcd}$$

Giving in homology les

$$\hookrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$\hookrightarrow H\ker(d'_{n-1}) \longrightarrow \cancel{HP'_{n-1}}^{\,0} \longrightarrow \cancel{H\ker d'_{n-2}}^{\,0}$$

$$\hookrightarrow H\ker(d_{n-1}) \longrightarrow \cancel{HP_{n-1}}^{\,0} \longrightarrow \cancel{H\ker(d_{n-2})}^{\,0}$$

$$\hookrightarrow H\ker(d''_{n-1}) \longrightarrow \cancel{HP''_{n-1}}^{\,0} \longrightarrow \cancel{H\ker(d''_{n-2})}^{\,0}$$

$$\hookrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$\Rightarrow H\ker(d''_{n-1}) \cong 0.$$

$$\Rightarrow$$

For $F : \underline{A} \longrightarrow \underline{B}$ there is a cannonical natural iso $R^0 F \cong F$.

F is left exact & $\underline{A}$ has enough injectives to talk about RF.

So let $A \longrightarrow I_A^\bullet$ be an injective res of some $A \in \underline{A}$.

Then $A \longrightarrow I_A^0$ is the kernel of $d : I_A^0 \longrightarrow I_A^1$

F preserves kernels so

$FA \longrightarrow FI_A^0$ is kernel of $Fd : FI_A^0 \longrightarrow FI_A^1$.

But because $I_A^\bullet$ is a resolution

$$F(H^0 I_A^\bullet) = F(\ker(d)) = \ker(Fd) \cong H^0(FI^\bullet_A) \cong R^0 F(A)$$

$\Rightarrow F(A) \cong R^0 F(A) \quad \forall A \in \underline{A}$

$\underline{\text{mod}}_R$ satisfies AB5.

AB3 + filtered colimits are exact.

Let $\underline{I}$ be a filtered category.

The colim functor is right exact. has a right adjoint.

$$\underline{\text{Mod}}_R^{\underline{I}} \longrightarrow \underline{\text{mod}}_R$$

In modules so $\ker(\alpha_i) = 0$.

So we need only to show it preserves kernels. (or monomorphisms).

Let $\alpha : F \longrightarrow G$ a natural transformation s.t. $\forall i \in I$ $\alpha_i : Fi \to Gi$ is mono.

Let $x \in \ker(\operatorname{colim} \alpha) \subseteq \operatorname{colim}(F)$. $\operatorname{colim} \alpha : \operatorname{colim} F \longrightarrow \operatorname{colim} G$.

Pick an $i \in \underline{I}$ $\tilde{x} \in F(i)$ representing x. $(\tilde{x} + R = x)$.

$\underline{I}$ filtered so for $, F : \underline{I} \to \underline{\text{mod}}_R$ is

$$\operatorname{Colim} F = \bigoplus_{i \in \underline{I}} F(i) \Big/ R$$

R is submodule generated by $\{m_i - f_*(m_i) : \forall f : i \to j, m_i \in F(i)\}$

why...

$$\operatorname{colim} \alpha_i(x) = 0 \Rightarrow \exists\, i \xrightarrow{f} j \quad f_*(\alpha_i(\tilde{x})) = 0 \in G(j).$$

rest follows from monomorphicity.

$F : \underline{Mod}_R \longrightarrow \mathcal{A}$ a left adjoint. $\mathcal{A}$ satisfies AB4. I a set & $\{ M_i \in \underline{Mod}_R : i \in I \}$

$$\Rightarrow \bigoplus_{i \in I} L_s F(M_i) \xrightarrow{\sim} L_s F(\bigoplus_{i \in I} M_i)$$

$\underline{Mod}_R$ has enough projectives so for each M_i take resolutions

$$P_{i,\bullet} \longrightarrow M_i$$

By AB3 all set indexed colimits exist moreover by AB4 for $\underline{mod}_R \Rightarrow \bigoplus$ is exact

Then $\bigoplus_i P_{i,\bullet} \longrightarrow \bigoplus_i M_i$ is a proj res.

$$\Rightarrow \bigoplus_i L_s F(M_i) = \bigoplus H_s F(P_{i,\bullet})$$
$$\xrightarrow{\sim} H_s \bigoplus F(P_{i,\bullet}) \qquad \bigoplus \text{ Exact}$$
$$\xrightarrow{\sim} H_s F(\bigoplus_i P_{i,\bullet}) \qquad F \text{ ~~left~~ adjoint}$$
$$= L_s F(\bigoplus_i M_i)$$

∎

AB4 ⇒ $\bigoplus$ exact ?

Colim is ~~left~~ right exact (adjoint) AB4 gives the other exactness?

Think so

k commutative ring. M a k-module.
$N : \underline{I} \longrightarrow \underline{\mathrm{mod}}_k$ a functor on a filtered small category.

$\Rightarrow \forall s \geq 0$

$\operatorname{colim} \operatorname{Tor}_s^k(M, N_i) \cong \operatorname{Tor}_s^k(M, \operatorname{colim} N_i)$

Tor is balanced so take a proj-res of M

$$P_\bullet \xrightarrow{\varepsilon} M.$$

Then $P_t \otimes -$ is a left adjoint so (commutes with colim.)

$$\operatorname{colim}(P_i \otimes N_i) \cong P_\bullet \otimes (\operatorname{colim} N_i)$$

Thus in homology

$$H_s(\operatorname{colim} P_\bullet \otimes N_i) \cong H_s(P_\bullet \otimes \operatorname{colim} N_i)$$
$$= \operatorname{Tor}_s^k(M, \operatorname{colim} N_i)$$

AB5 filtered colimits exact.

$$\operatorname{colim} \operatorname{Tor}_s^k(M, N_i) = \operatorname{colim} H_s(P_\bullet \otimes N_i) \cong H_s(\operatorname{colim} P_\bullet \otimes N_i)$$

$\forall A, B \in \underline{Ab}$
$\operatorname{Tor}_s^{\mathbb{Z}}(A, B) = 0$
$s > 1$

Choose a free abelian group F_0 and epi

$$F_0 \xrightarrow{\varepsilon} B$$

The kernel of ε is free
$\Rightarrow$ kernel is projective

Subgroups of free groups are free.

So $0 \longrightarrow \ker(\varepsilon) \longrightarrow F_0 \xrightarrow{\varepsilon} B \longrightarrow 0$
a projective resolution of B.

An abelian group is flat iff it is torsion free.

($\Rightarrow$) Suppose A is flat.
Consider $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$
an injective resolution of $\mathbb{Z}$.

Apply $\mathrm{Tor}^{\mathbb{Z}}(A,-)$ to get a long exact sequence.

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Tor}_0(A,\mathbb{Z}) \to \mathrm{Tor}_0(A,\mathbb{Q}) \to \cdots$$

$$\equiv \quad \cdots \to \underbrace{\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q})}_{\cong 0} \to \underbrace{\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})}_{\cong A_{tor}} \to \underbrace{A \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong A} \rightarrowtail A \otimes_{\mathbb{Z}} \mathbb{Q} \to A \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

mono b/c A is flat

So $0 \to A_{tor} \to A \rightarrowtail A \otimes \mathbb{Q}$ is exact

$\Rightarrow A_{tor} = 0$ b/c $A_{tor} \to A$ is ker of $A \rightarrowtail A \otimes \mathbb{Q}$ by exactness which is zero by mono.

($\Leftarrow$) Let A be a torsion free abelian group.
Every finitely generated subgroup $A' \subseteq A$ is torsion free hence free

$\Rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A',-) = 0$

Recall $A = \operatorname{colim}_{\substack{A' \subseteq A \\ \text{fin gen}}} A'$

$\Rightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A,-) = \operatorname{colim} \mathrm{Tor}_1^{\mathbb{Z}}(A',-) = 0$ (Tor commutes with colim)

Let $0 \to B' \to B \to B'' \to 0$ be a ses.
Get a les in Tor

$$\cdots \to \underbrace{\mathrm{Tor}_1^{\mathbb{Z}}(A,B')}_{\cong 0} \to A \otimes_{\mathbb{Z}} B' \to A \otimes_{\mathbb{Z}} B \to A \otimes_{\mathbb{Z}} B'' \to 0$$

$$\equiv \quad 0 \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

is a ses

$\Rightarrow A$ is flat.

R a commutative ring.
$M \in \underline{\text{mod}}_R$ flat

$\Leftrightarrow \forall I \trianglelefteq R$ (ideals)
$\mathrm{Tor}_1^R(R/I, M) = 0$

($\Rightarrow$) let M be flat.

$$\Rightarrow (- \underset{R}{\otimes} M)(I \hookrightarrow R)$$

$$\cong I \underset{R}{\otimes} M \rightarrowtail R \underset{R}{\otimes} M \cong M$$

$$\Rightarrow \mathrm{Tor}_1^R(R, M) \longrightarrow \mathrm{Tor}_1^R(R/I, M) \longrightarrow 0 \quad ☆$$

in the Tor les of $0 \to I \to R \to R/I \to 0$

$\mathrm{Tor}_1^R(R,M) = 0$ b/c R is a proj R module.

$$\Rightarrow \mathrm{Tor}_1^R(R/I, M) \cong 0.$$

☆

$$\mathrm{Tor}_0^R(I, M) \longrightarrow \mathrm{Tor}_0^R(R, M) \longrightarrow \mathrm{Tor}_0^R(R/I, M)$$

$$\equiv \underbrace{I \otimes M \rightarrowtail}_{\ker 0} M \longrightarrow R/I \otimes M$$

$$\Rightarrow \mathrm{Im}(\delta) = 0.$$

boundary from les

($\Leftarrow$) let $M \in \underline{\text{Mod}}_R$ such that $\forall I \trianglelefteq R$ $\mathrm{Tor}_1^R(R/I, M) = 0$.

Again from les $0 \longrightarrow \mathrm{Tor}_0^R(I, M) \longrightarrow \mathrm{Tor}_0^R(R, M)$

$\equiv 0 \longrightarrow I \otimes M \longrightarrow R \underset{R}{\otimes} M \cong M$ is exact.

We know that $- \otimes M$ is right exact so we only need to show left exactness. So we show that $- \otimes M$ preserves monomorphisms:

$0 \to K \xrightarrow{g} R \xrightarrow{f} Q$

($\Rightarrow$) monomorphisms
($\Leftarrow$) kernels
show

So let $Q \rightarrowtail N$ be mono.

Consider the poset

Seemingly a monomorphism is an inclusion of a kernel...

$(S, \leq) \to (T, \subseteq) \xrightarrow{\text{Trivial}} Mod_R$

$\quad\quad\quad\quad\;\; \cong$

$\quad\quad (Sub(M), \subseteq)$

colim here union

$$\mathcal{S} = \left\{ Q' : \begin{array}{l} Q \subseteq Q' \subseteq N \\ Q \otimes M \longrightarrow Q' \otimes M \quad \text{is mono} \end{array} \right\}$$

ordered by inclusion.
Note it is nonempty b/c $Q \in \mathcal{S}$.

If we can show that $N \in \mathcal{S}$ then we would be done.

when is a colimit a union?

So we proceed:
let $\mathcal{S}' \subseteq \mathcal{S}$ a chain (a totally ordered subset)

$$\Rightarrow Q_{\mathcal{S}'} = \bigcup_{Q' \in \mathcal{S}'} Q'$$

$$= \operatorname{colim}_{\mathcal{S}'} Q'$$

why

i.e. $Q_{\mathcal{S}'} \in \mathcal{S}$

$$\Rightarrow Q \underset{R}{\otimes} M \longrightarrow Q_{\mathcal{S}'} \underset{R}{\otimes} M = \operatorname{colim}_{\mathcal{S}'} (Q' \otimes M)$$

is a monomorphism b/c $\otimes$ commutes with colimits & modules satisfies AB5 (Directed colimit of monos is mono)

$$\operatorname{colim}_{\mathcal{S}'} Q \underset{R}{\otimes} M = Q \otimes M.$$

So $Q_{\mathcal{S}'} \in \mathcal{S}$ and is an upper bound of $\mathcal{S}'$

every chain has upper bound $\Rightarrow \exists$ maximal element.

Apply Zorns lemma to get $Q_{max} \in \mathcal{S}$

Now claim $Q_{max} = N$.

For a contradiction assume $Q_{max} \neq N$.

$\Rightarrow \exists x \in N \setminus Q_{max}$

let $\tilde{Q} = \underbrace{Q_{max} + Rx}$ ← submodule generated by x.

Both submoduals of N. + means $A + B = \{a + b : a \in A, b \in B\}$

I = annihilator(x)

Then $0 \to Q_{max} \longrightarrow \tilde{Q} \longrightarrow \tilde{Q}/Q_{max} \longrightarrow 0$ is ses.

By hypothesis? $\tilde{Q}/Q_{max}$ is generated by $x + Q_{max}$ thus $\exists I \trianglelefteq R$

$$\tilde{Q}/Q_{max} \cong R/I \;?$$

Then by assumption

$$\mathrm{Tor}_1^R(\tilde{Q}/Q_{max}, M) = 0.$$

Thus in long exact sequence we get (degree 0)

$$0 \longrightarrow Q_{max} \otimes M \longrightarrow \tilde{Q} \otimes M$$

$\Rightarrow \tilde{Q} \in \mathcal{S}$ & $Q_{max} \subset \tilde{Q}$
Contradicting maximality of Q_{max}
$\rightarrow\leftarrow$

$\Rightarrow Q_{max} = N$

$\Rightarrow N \in \mathcal{S}$

$\Rightarrow - \otimes M$ preserves monomorphisms
$\Rightarrow$ left exact
$\Rightarrow$ flat. $\square$

$A \rightarrowtail B \twoheadrightarrow C$

$\boxed{\text{A sequence } \xi \text{ splits} \Leftrightarrow \Theta(\xi) = 0}$

$\Theta(\xi) = 0$

$\Leftrightarrow \delta(id_A) = 0$

$\Leftrightarrow id \in \ker(\delta) = im(\, Hom(A,E) \to Hom(A,A))$

$\Leftrightarrow \exists s : A \to E$ such that

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[rr, bend right, "id"'] & E \arrow[r] & A \end{tikzcd}$$

G

$\Leftrightarrow \xi$ splits (splitting lemma).

$\Theta : \{\text{Classes of extensions of } A \text{ by } B\} \xrightarrow{\sim} Ext^1(A,B)$

Let I be injective & $0 \to B \xrightarrow{i} I \xrightarrow{p} M \to 0$ a ses.

Get les in $Ext(A,-)$

$$\to Hom(A,I) \to Hom(A,M) \xrightarrow{\delta_{i,p}} Ext^1(A,B) \to Ext^1(A,I) \cong 0.$$

b/c I is injective so $0 \to I \to 0$ as resolution $Ext^1(A,I) \cong Hom(I,0) \cong 0$.

$\Rightarrow \delta_{i,p}$ is epimorphism of groups thus surjective.

Long & unfinished in class.

$(-)^G : \underline{\mathrm{mod}}_G \longrightarrow \underline{Ab}$
$M \longmapsto \{m \in M : \forall g \; gm = m\}$
is *left* exact functor

Denote $\mathbb{1}$ be the trivial G-module $\mathbb{Z}$.
(module over group ring $\mathbb{Z}G$ s.t. $\forall g \in G \; \forall n \in \mathbb{Z}$
$g \cdot n = n$)

Then $\mathrm{Hom}_G(\mathbb{1}, M) \xrightarrow{\sim} M^G$
$\varphi \longmapsto \varphi(1)$

is a natural isomorphism b/c
$\forall \varphi \in \mathrm{Hom}_G(\mathbb{1}, M)$ φ is just a choice of where to send $1 \in \mathbb{Z}$ to in M
but $\mathbb{Z}$ must act trivially so it must choose an element in M^G
(otherwise it would fail to be a homomorphism)
$\varphi(n) = \varphi(g \cdot n) = g \cdot \varphi(n) = \varphi(n)$
So $\forall g$ $g \cdot \varphi(n) = \varphi(n)$
$\forall n$ $\varphi(n) \in M^G$.

Hom is a left exact functor. $\square$

$\underline{\mathrm{Mod}}_G \longrightarrow \underline{Ab}$
$M \longmapsto M_G$
$M_G = M / (m \sim m' \Leftrightarrow \exists g \; gm = m')$
is right exact.

Because we evaluate equality component wise this functor is clearly additive.
Need to show preservation of cokernels or whats the same preservation of epimorphisms.

So let $M \xrightarrow{f} N$ be a $\underline{\mathrm{mod}}_G$ epi.

First what does M_G do on morphisms?
$(f : M \to N) \longmapsto \left(\begin{array}{l} f_G : M_G \longrightarrow N_G \\ \quad [x] \longmapsto [fx] \end{array} \right)$

Need to check thats well defined.

Then preservation of epimorphisms is clear.

G a finite group
$N = \sum_{g \in G} g$ the norm element (of $\mathbb{Z}G$)

- $N \in (\mathbb{Z}G)^G$
- $(\mathbb{Z}G)^G = \mathbb{Z}N$
- $N^2 = |G| \cdot N$

i) let $h \in G$ $\quad hN = h \sum_g g$
$$= \sum_g hg$$
$$= N$$

ii) let $x = \sum_{g \in G} a_g g \in (\mathbb{Z}G)^G$

Then $\forall h \in G \quad hx = x$
$$\Rightarrow x = \sum a_g g = \sum a_g (hg) = \sum a_{h^{-1}g}\, g$$
$$\Rightarrow a_g = a_{h^{-1}g} \quad \forall g$$
$$\Rightarrow \forall g \quad a_g = a_e$$
$$\Rightarrow x = a_e \cdot N.$$

iii) $N^2 = \left(\sum g\right) N = \sum gN = \sum_{g \in G} N = |G| N$

Let k be a commutative ring such that $s = |G|$ is invertible in k.

M a kG module
$\Rightarrow H_0(G,M) \cong H^0(G,M) \cong \frac{N}{s} M$
& $H_n(G,M) \cong H^n(G,M) \cong 0$
$n \geq 1$

Think about s^{-1} here & what happens when its not inv in k.

First show cohomology.

Recall $H_n(G,M) = L_n(-)_G(M)$
$\cong \mathrm{Tor}^G_n(\mathbb{Z}, M)$.
why

& $H^n(G,M) = R^n(-)^G(M)$
$\cong \mathrm{Ext}^n_G(\mathbb{Z}, M)$

why define like this

First $H^0(G,M) = M^G$ b/c $R^0(-)^G(M) \cong M^G$.
So we show $\frac{N}{s} M = M^G$.

$s^{-1} N M \subseteq M^G$ by previous theorem.

Let $x \in M^G \Rightarrow s^{-1} N x = s^{-1} \sum gx$
$$= s^{-1} \sum x$$
$$= s^{-1}(sx)$$
$$= x.$$

$\Rightarrow x \in s^{-1} N M$.

Group homology, Tor, Ext, Ind { relation?

Then
Homology

So $s^{-1}NM = M^G = H^0(G,M)$.

$\mathrm{Ind}_H^G : \underline{\mathrm{mod}}_H \to \underline{\mathrm{mod}}_G$ is exact & left adjoint to Res_G^H.

Recall:

$\mathrm{Ind}_H^G : \underline{\mathrm{mod}}_H \longrightarrow \underline{\mathrm{mod}}_G$ let $H \leq G$.

$M \longmapsto \mathbb{Z}G \otimes_{\mathbb{Z}H} M$

$\mathrm{Res}_G^H : \underline{\mathrm{mod}}_G \longrightarrow \underline{\mathrm{mod}}_H$

$M \longmapsto M$ (b/c $H \leq G$ M is also a $\mathbb{Z}H$ module with action inherited by H as a subgroup).

Now: $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module on a set of coset representatives of H in G. In particular it is flat

$\Rightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} -$ is exact

Adjunction comes from an exercise.

$\mathrm{CoInd}_H^G : \underline{\mathrm{mod}}_H \longrightarrow \underline{\mathrm{mod}}_G$

$M \longmapsto \mathrm{Hom}(\mathbb{Z}G, M)$

$(g \cdot \varphi)(x) = \varphi(g^{-1} \cdot x)$

is right adjoint to restriction & exact.

Exactness feels out of scope.

Shapiros Lemma:
$H \leq G$. $M \in \underline{mod}_H$

$\Rightarrow H_*(G, \mathrm{Ind}_H^G M) \cong H_*(H, M)$
$H^*(G, \mathrm{CoInd}_H^G M) \cong H^*(H, M)$

Proof

i) Homology:
Recall Ind is exact & a left adjoint to an exact functor.

Let $P_\bullet \to M$ be a $\underline{mod}_H$ projective resolution of M.

By exactness $\mathrm{Ind}_H^G(P_\bullet) \longrightarrow \mathrm{Ind}_H^G(M) \longrightarrow 0$ is a left resolution as $\mathbb{Z}G$ modules.
but it has an exact right adjoint so it preserves projectives. i.e. This is a $\mathbb{Z}G$ projective resolution of $\mathrm{Ind}_H^G(M)$

For any H module N

Recall

$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$

$$\begin{tikzcd}
N_H \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Ind}_H^G(N)_G \arrow[d, equal] \\
\mathbb{Z} \otimes_{\mathbb{Z}H} N \arrow[r, "\sim"] & \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} N
\end{tikzcd}$$

$$\Rightarrow (P_\bullet)_H \xrightarrow{\sim} (\mathrm{Ind}_H^G P_\bullet)_G$$

$$\Rightarrow H_*(H, M) \cong H_*((P_\bullet)_H) \cong H_*(\mathrm{Ind}_H^G P_\bullet)_G = H_*(G, \mathrm{Ind}_H^G(M))$$

ii) Cohomology is the same.

$\swarrow$ finite # of cosets

$[G:H]<\infty$
$\Rightarrow \mathrm{Ind}_H^G \cong \mathrm{CoInd}_H^G$.

WHY

Let $\{g_\alpha\}$ be a set of [coset reprentatives] ? of H in G.

$\{$ $\{g_\alpha\}$ forms a $\mathbb{Z}H$ basis of $\mathbb{Z}G$. $\}$

The following is G-equivariant for some H-module

$$\mathrm{Ind} \rightarrow \mathbb{Z}G \otimes_{\mathbb{Z}H} M \xrightarrow{\varphi} \mathrm{Hom}_H(\mathbb{Z}G, M) \leftarrow \mathrm{CoInd}$$

$$g_\alpha \otimes m \longmapsto f_{\alpha,m}$$

Every object is a linear combination of pure tensor

Kronecker.

$$g_\beta \longmapsto m\,\delta_{\alpha\beta}$$

$\cong \mathrm{maps}(\{g_\alpha\}, M)$

which is the map

$$\bigoplus_\alpha M \xrightarrow{\psi} \prod_\alpha M$$

a cannonical iso b/c $[G:H]<\infty$.

So $\varphi=\psi \Rightarrow \mathrm{Ind} \cong \mathrm{CoInd}$.

A section $G \to E$.

A group extension is split iff

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\varphi", "\wr"'] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A \arrow[r] & A \rtimes G \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

($\Longleftarrow$) Given iso φ

$$G \longrightarrow E$$
$$g \longmapsto \varphi(0, g)$$ is a section $\square$

($\Longrightarrow$) Suppose $0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 0$ is split.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & E \arrow[r, "\pi"] & G \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 0 \end{tikzcd}$$

Define $\varphi : A \rtimes G \longrightarrow E$
$$(a, g) \longmapsto i(a)\sigma(g)$$

which is a group iso.